Odd and Even Functions

Before we introduce hyperbolic functions, we need to understand odd and even functions.

Fact (Even and Odd Functions) — A function $f(x)$ is:

- **even** if $f(-x) = f(x)$ for all x in its domain
- **odd** if $f(-x) = -f(x)$ for all x in its domain

Example

$\sin(x)$ is an odd function

Theorem (Decomposition into Odd and Even Functions)

Any function f with a symmetric domain can be written as the sum of an even function g and an odd function h:

$$f(x) = g(x) + h(x)$$

Example

Express $f(x) = e^x$ as the sum of an even and an odd function.

Hyperbolic Function Definitions

Fact (Hyperbolic Sine and Cosine) — The hyperbolic functions are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Note that $\cosh x$ is even and $\sinh x$ is odd.

Tip

The notation comes from adding ‘h’ (for ‘hyperbolic’) to the trigonometric symbols. Pronounce sinh as “shine”, and cosh as “cosh”.

Example

Evaluate (a) $\cosh(0)$, (b) $\sinh(0)$, (c) $\cosh(\ln 2)$, (d) $\sinh(\ln 3)$

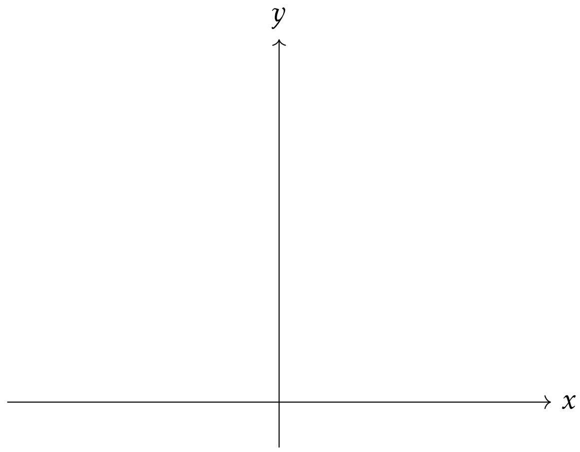
Example

What is $\tanh x$?

Graphs of Hyperbolic Functions

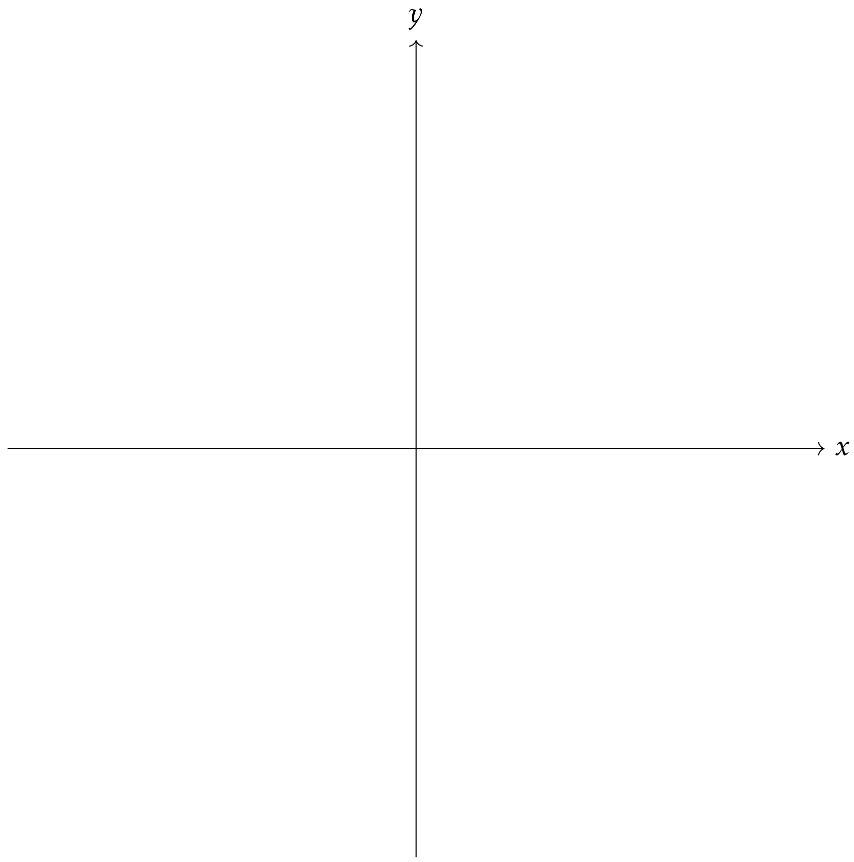
Example

Sketch $y = \cosh x$



Example

Sketch $y = \sinh x$

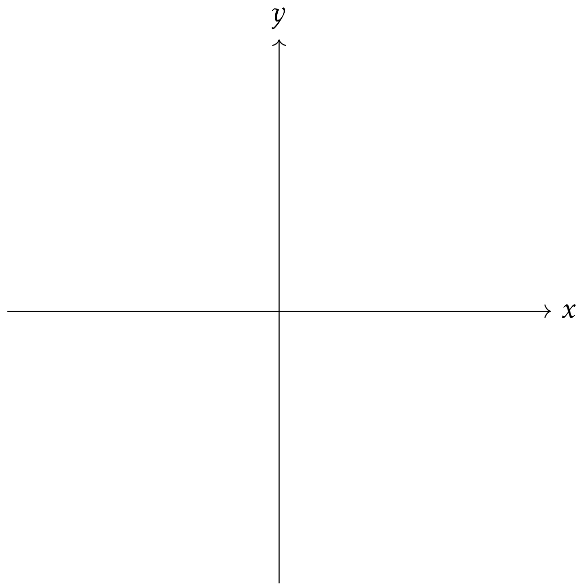


Fact (Key Properties of cosh, sinh and tanh x) —

- $\cosh x \geq 1$ for all x, with minimum at $x = 0$
- $\sinh x, \tanh x$ passes through the origin and is strictly increasing
- As $x \to \infty$: $\cosh x \approx \sinh x \approx \frac{1}{2}e^x$, $\tanh x \approx 1$
- As $x \to -\infty$: $\cosh x \approx \frac{1}{2}e^{-x}$, $\sinh x \approx -\frac{1}{2}e^{-x}$, $\tanh x \approx -1$

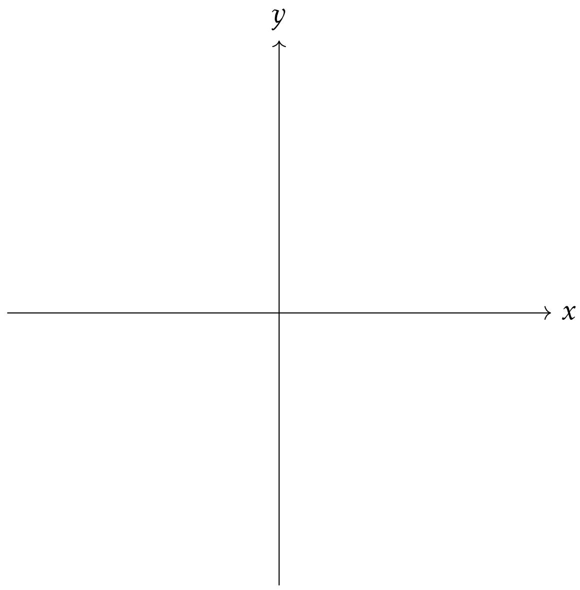
Example

Sketch $y = 2\sinh(x+2)$



Example

Sketch $y = \text{sech}(x)$



Hyperbolic Identities

Example

Prove that $\cosh^2 x - \sinh^2 x \equiv 1$

Tip

Compare with $\cos^2 x + \sin^2 x = 1$. Note the sign difference!

Fact (Addition Formulae) —

$$\begin{aligned}
\sinh(A+B) &= \sinh A \cosh B + \cosh A \sinh B \\
\sinh(A-B) &= \sinh A \cosh B - \cosh A \sinh B \\
\cosh(A+B) &= \underbrace{\cosh A \cosh B + \sinh A \sinh B}_{\text{notice the sign difference}} \\
\cosh(A-B) &= \underbrace{\cosh A \cosh B - \sinh A \sinh B}_{\text{notice the sign difference}}
\end{aligned}$$

Example

Prove that $\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$

Fact (Double Angle Formulae) —

$$\cosh 2A = 2\cosh^2 A - 1 = \underbrace{\cosh^2 A + \sinh^2 A}_{\text{notice the sign difference}} = \underbrace{1 + 2\sinh^2 A}_{\text{notice the sign difference}}$$

$$\sinh 2A = 2\sinh A \cosh A$$

Tip (Osborn's Rule)

When converting a trignometric identity to a hyperbolic trig identity, “stick an ‘h’ on the end” and whenever two sins are multiplied together, flip the sign

Example

Prove that $\cosh^4 x \equiv \frac{1}{8}\cosh 4x + \frac{1}{2}\cosh 2x + \frac{3}{8}$

Example

Prove that $\sinh 5x \equiv 16\sinh^5 x + 20\sinh^3 x + 5\sinh x$

Differentiating hyperbolic trig functions

Example

Find $\frac{d}{dx}(\cosh x)$ and $\frac{d}{dx}(\sinh x)$

Example

Find $\frac{d}{dx}(\tanh x)$

Maclaurin Series

Fact (Maclaurin Series for e^x) — From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Example

Find the Maclaurin series for $\cosh x$ and $\sinh x$

Tip

Note that $\cosh x$ has only even powers (it's an even function) and $\sinh x$ has only odd powers (it's an odd function). All coefficients are positive!

Example

Find the Maclaurin series for $\tanh x$ up to the x^5 term.

Inverse Hyperbolic Functions

Example

Find a formula for $\sinh^{-1} x$

Example

Find a formula for $\tanh^{-1} x$